Pergamon

PII: S0021-8928(98)00004-5

*J. Appl. Maths Mechs*, Vol. 62, No. 1, pp. 37–42, 1998

0021-8928/98 $24.00+0.00

# STABILITY REGIONS OF LINEAR CANONICAL SYSTEMS WITH PERIODIC COEFFICIENTS†

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(*Received* 10 *June* 1996)

A new definition of the index of the stability region of a canonical system of linear differential equations with periodic coefficients is proposed. A simple proof of the Gel'fand–Lidskii theorem [1] on the structure of stability regions is given and a theorem on the directional convexity of such regions is proved. It follows from this theorem, in particular, that stability regions of parametric oscillations in a system with a sign-definite Hamiltonian are convex with respect to the frequency of parametric perturbation.


## 1. FORMULATION OF THE PROBLEM

Consider the canonical system

$$J\dot{\mathbf{x}} = H(t)\mathbf{x}, \quad J = \begin{Vmatrix} 0 & -I_n \\ I_n & 0 \end{Vmatrix}, \quad \mathbf{x} \in R^{2n} \tag{1.1}$$

where $H(t) = H(t + T)$ is a symmetrical piecewise-continuous matrix (a Hamiltonian) of order $2n$ and $I_n$ is the identity matrix of order $n$.

Problems of parametric oscillations and of the dynamic stability of elastic systems, problems of the stability of periodic oscillations of non-linear Hamiltonian systems and many others reduce to Eq. (1.1). The fundamentals of the theory of such equations were laid by Lyapunov and Poincaré; later, Krein and others obtained many profound results (a systematic description of the theory can be found in [2, 3]).

System (1.1) is said to be stable if all its solutions are bounded as $t \to \infty$. Strongly stable systems are of practical interest. In such systems, stability is preserved for fairly small perturbations of the matrix $H(t)$, without disturbing its symmetry.

We will present some well-known facts which will be used later. The matrix of the fundamental system of solutions of Eq. (1.1) $X(t) = [\mathbf{x}_p(t)]_1^{2n}$ satisfies the identity

$$X(t)^*(iJ)X(t) = C \tag{1.2}$$

where $C = [c_{pk}]_1^{2n}$ is a constant matrix.

Henceforth we will use the inverse assertion [1]: if a certain matrix $X(t)$ satisfies identity (1.2), then $X(t)$ is a solution of Eq. (1.1) with Hamiltonian

$$H(t) = J\dot{X}(t)X^{-1}(t) \tag{1.3}$$

Using the classification introduced by Krein [4] the multipliers of Eq. (1.1) can be divided into multipliers of the first and second kind. For strong stability it is necessary and sufficient [1, 4], that all the multipliers should lie on the unit circle and be definite, i.e. there should not be any identical multipliers of different kinds among them.

In a stable system, the solutions can be represented in the form

$$\mathbf{x}_p(t) = \exp(i\omega_p t)\mathbf{f}_p(t), \quad \mathbf{x}_{p+n}(t) = \mathbf{x}_p^*(t), \quad p = 1,\ldots,n \tag{1.4}$$

where $\mathbf{f}_p(t)$ are $T$-periodic functions, and $i\omega_p = T^{-1}\ln \rho_p$ are characteristic factors. Henceforth we will

†*Prikl. Mat. Mekh.* Vol. 62, No. 1, pp. 41–48, 1998.

assume that $0 < \omega_p < 2\pi$, $\rho_p$ and $\rho_{p+n} = \rho_p^*$ ($p = 1, \ldots, n$) are multipliers of the first and second kind, respectively.

For solutions (1.4)

$$c_{pk} = (\mathbf{x}_p(t), iJ\mathbf{x}_k(t)) = [(\exp(i\omega_p - i\omega_k)t]((\mathbf{f}_p(t), iJ\mathbf{f}_k(t)) \quad (1.5)$$

where $(\mathbf{a}, \mathbf{b})$ is the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

Taking into account the periodicity of the functions $\mathbf{f}_q(t)$ we obtain $c_{pk} = 0$ when $\omega_p \neq \omega_k$. For simple multipliers of the first and second kind $c_{pp} > 0$ and $c_{pp} < 0$ respectively [4]; without loss of generality we will assume $|c_{pp}| = 2$. In the case of an $m$-fold multiplier, the functions $\mathbf{f}_p(t)$ ($p = 1, \ldots, m$) corresponding to it can be chosen in such a way that the equality $c_{pk} = 0$ when $p \neq k$ remains true.

Assuming $\mathbf{f}_p(t) = \mathbf{u}_p(t) + i\mathbf{v}_p(t)$ ($p = 1, \ldots, n$), we obtain $c_{pp} = 2(J\mathbf{u}_p(t), \mathbf{v}_p(t))$. Hence, the real functions $u_p(t)$ and $v_p(t)$ satisfy the relations

$$(J\mathbf{u}_p, \mathbf{u}_p) = 0, \quad (J\mathbf{v}_p, \mathbf{v}_p) = 0, \quad (J\mathbf{u}_p, \mathbf{v}_k) = 0, \quad p \neq k; \quad (J\mathbf{u}_p, \mathbf{v}_p) = 1 \quad (1.6)$$

## 2. THE STRUCTURE OF THE STABILITY REGIONS

In a strongly stable system, on going round the unit circle groups of multipliers of different order are encountered in sequence. The multiplier type of such a Hamiltonian can be defined by the set of numbers $n_p$ ($p = 1, \ldots, r$), where $r$ is the number of such groups on the upper semicircle, $|n_p|$ is the number of multipliers in the $p$th group, $n_p > 0$ and $n_p < 0$ for multipliers of the first and second kind, respectively and $|n_1| + \ldots + |n_r| = n$.

Two strongly stable Hamiltonians $H_1(t)$ and $H_2(t)$ belong to one stability region if a Hamiltonian $H(t, s) = H(t + T, s)$, continuous in $s$, exists such that $H(t, 0) = H_1(t)$, $H(t, 1) = H_2(t))$ and the corresponding Eq. (1.1) is strongly stable when $s \in [0, 1]$; otherwise $H_1(t)$ and $H_2(t)$ belong to different stability regions. By the Gel'fand–Lidskii theory [1] each stability region is uniquely defined by the multiplier type and one integer $k$ ($-\infty < k < \infty$), called the index of rotation or simply the system index.

We will put $L(t) = [\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t), \mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)]$. By virtue of (1.6)

$$L(t)^* JL(t) = J \quad (2.1)$$

i.e. $L(t)$ is a simplectic matrix. Suppose $\mathrm{Sp}(2n)$ is a set (group) of such matrices of order $2n$; then $L(t) \in \mathrm{Sp}(2n)$ is a closed curve ($L(0) = L(T)$).

Following the well-known approach [5, p. 360], we will show that by continuous deformation in the group $\mathrm{Sp}(2n)$ it may be contracted to the matrix $L_0 = [l^0_{pq}]_i^{2n}$, the non-zero elements of which are

$$l^0_{pp} = 1, \quad p \neq n, \quad 2n, \quad (2.2)$$

$$l^0_{nn} = l^0_{2n,2n} = \cos(2\pi kt / T), \quad l^0_{2n,n} = -l^0_{n,2n} = \sin(2\pi kt / T)$$

where $k$ is a certain integer, which will also be called the index of system (1.1).

In fact, since any non-zero vector is the first column of a certain matrix $L \in \mathrm{Sp}(2n)$, then $\mathbf{u}_1(t)$ may contract to the point $(1, 0, \ldots, 0)$. Here the matrix $L(t)$ is transformed into $L^1(t)$, the elements of the $(n + 1)$th row of which, by virtue of (2.1), are $l^1_{n+1,n+1} = 1$, $l^1_{n+1,q} = 0$, $q \neq n + 1$. Obviously, for this first column, the elements $l_{q,n+1}$, $q \neq n + 1$ may take any values in the group $\mathrm{Sp}(2n)$, and hence they may contract to zero. As a result, taking (2.1) into account we obtain $l_{1q} = l_{q1} = 0$, $q > 1$, $l_{q,n+1} = l_{n+1,q} = 0$, $q \neq n + 1$ for the matrix $L^2(t)$; hence, the initial curve $L(t)$ is confined in the cycle to the subgroup $\mathrm{Sp}(2n-2)$. Repeating this procedure $n - 1$ times, we find that the matrix obtained differs from $L_0$ solely by the elements $l_{nn}(t)$, $l_{n,2n}(t)$, $l_{2n,n}(t)$ and $l_{2n,n}(t)$. Taking into account the fact that its determinant is equal to $l_{nn}(t)l_{2n,2n}(t) - l_{n,2n}(t)l_{2n,n}(t) = 1$, it can be shown that these elements contract to the form (2.2).

Suppose $H_1(t)$ and $H_2(t)$ are strongly stable Hamiltonians with the same multiplier type, and $k_1$ and $k_2$ are the corresponding indices. The following theorem is analogous to the Gel'fand–Lidskii theorem [1], but the proof given below is considerably simpler.

*Theorem* 1. The Hamiltonians $H_1(t)$ and $H_2(t)$ belong to one region of stability if only $k_1 = k_2$.

*Proof*. Suppose $H(t, s)$ is a continuous stable curve, connecting the Hamiltonians $H_1(t)$ and $H_2(t)$ ($s \in [0, 1], H(t, 0) = H_1(t), H(t, 1) = H_2(t)$). The fundamental system of solutions (1.4) has the form

$$X(t, s) = F(t, s)E(t, s) \tag{2.3}$$

$$F(t,s) = F(t+T,s) = [\mathbf{f}_p(t,s)]_1^{2n}, \quad E(t,s) = \mathrm{diag}[\exp(i\omega_p(s)t], \quad p = 1,\ldots,2n$$

$$F(t,0) = F^1(t), \quad F(t,1) = F^2(t), \quad \mathbf{f}_p(t,s) = \mathbf{u}_p(t,s) + i\mathbf{v}_p(t,s)$$

The matrix $L(t, s) = [\mathbf{u}_p(t, s), \mathbf{v}_p(t, s)]_1^n$ is continuous over $s$ and converts $L^1(t) = [\mathbf{u}_p^1(t), \mathbf{v}_p^1(t)]_1^n$ into $L^2(t) = [\mathbf{u}_p^2(t), \mathbf{v}_p^2(t)]_1^n$. It is obvious that for continuous deformation of the matrix $L(t, s)$ its index $k(s)$ is preserved, and hence $k(0) = k_1 = k(1) = k_2$, which also proves the necessity of the condition $k_1 = k_2$.

For the proof it is sufficient to construct the required Hamiltonian $H(t, s)$. As was shown above, when $k_1 = k_2 = k$ the matrices $L^1(t)$ and $L^2(t)$ may be contracted into $L_0$; consequently, a simplectic matrix $L(t, s)$ exists such that $L(t, 0) = L^1(t), L(t, 1) = L^2(t)$. The corresponding matrix $F(t, s)$ converts $F^1(t)$ into $F^2(t)$.

Since, by the condition, the multiplier types of the Hamiltonians $H_1(t)$ and $H_2(t)$ are the same, $\omega_p^1$ and $\omega_p^2$ can be connected by the curves $\omega_p(s)$, ($\omega_p(0) = \omega_p^1$, $\omega_p(1) = \omega_p^2$, $\omega_{p+n}(s) = -\omega_p(s)$), so that

$$\exp(i\omega_k(s)T) \neq \exp(-i\omega_p(s)T), \quad k, p = 1,\ldots,n, \quad s \in [0,1] \tag{2.4}$$

For these $F(t, s)$ and $\omega_p(s)$ the matrix $X(t, s)$, defined by (2.3), satisfies identity (1.2) (here $C = 2\mathrm{diag}[I_n, -I_n]$). Consequently, $X(t, s)$ is the matrix of fundamental solutions of system (1.1) with Hamiltonian $H(t, s)$, defined by (1.3); it is obvious that $H(t, 0) = H_1(t), H(t, 1) = H_2(t)$. The corresponding multipliers of the first and second kind are: $\rho_p(s) = \exp[i\omega_p(s)T]$ and $\rho_{p+n}(s) = \exp[-i\omega_p(s)T]$; by virtue of (2.4) Eq. (1.1) is strongly stable when $s \in [0, 1]$. Hence, the required Hamiltonian has been constructed, which completes the proof of the theorem.

*Note* 1. For certain constraints on the matrix $A(t)$ [2] the non-canonical system of order $2n$

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \quad A(t) = A(t+T) \tag{2.5}$$

also has a reciprocal characteristic equation. For the strong stability of this system it is necessary and sufficient that all the multipliers should lie on the unit circle and be simple. The matrix of the system of fundamental solutions of Eq. (2.5) can also be represented in the form (2.3), where $F(t)$ is a non-singular matrix. The matrix $L(t)$ corresponds to it, and, without loss of generality, we can take $\det L(t) = 1$; then $L(t) \in SL(2n)$ where $SL(2n)$ is a group of unimodular linear transformations.

Using a procedure similar to that given above, the matrix $L(t)$ can be deformed to the form $L_0 = \mathrm{diag}[I_{2n-2}, M]$, where $M$ is a matrix with elements $m_{11} = m_{22} = \cos(2\pi kt/T)$, $m_{21} = -m_{12} = \sin(2\pi kt/T)$ and $k$ is an integer. Hence, the stable matrices $H_1(t)$ and $H_2(t)$ can be connected by a stable curve if only the corresponding numbers $k_1 = k_2$. Hence, the stability regions of system (2.5) with reciprocal characteristic equation are also defined by the index $k$.

## 3. DETERMINATION OF THE INDEX

The procedure described above cannot, of course, be used to calculate the index $k$. Some constructive determinations of the index were described in [1, 2]; they reduce to calculating the increment in $[0, T]$ of the argument of a certain complex-valued function, expressed using the matrisant of Eq. (1.1).

We will give a new definition of the index of the stability region. Consider the self-conjugate boundary-value problem

$$J\dot{\mathbf{x}} = [D + \lambda(H(t) - D)]\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}(T) \tag{3.1}$$

where $D = -2\pi rT^{-1}I_{2n}$, and the integer $r \geqslant 0$ is taken from the condition $H(t) > D$ (to do this it is obviously necessary and sufficient that $\beta_*(t) > -2\pi rT^{-1}$ when $t \in [0, T]$, where $\beta_*(t)$ is the least eigenvalue of the matrix $H(t)$). In particular, if $H(t)$ is a positive definite matrix, we can assume $D = 0$.

Since $H(t) - D > 0$, the eigenvalues $\lambda_1, \lambda_2, \ldots$ of problem (3.1) are real [2]. Suppose $N$ is the number of eigenvalues in (0, 1]; we will call the quantity $q = N/2 - m$ the index of the Hamiltonian $H(t)$. This quantity is an even number independent of $r$.

In fact, for integer $r$ problem (3.1) has $2n$ zero eigenvalues $\lambda_1, \ldots, \lambda_{2n} = 0$. It can be shown by the method of perturbations that $d\lambda_k/dr > 0$, and hence when $r$ increases continuously they shift to the interval (0, 1); they henceforth remain in it, since the number of eigenvalues $\lambda = 1$ is independent of $r$. Consequently, when $r$ is increased by one the number $N$ increases by $2n$, and as a result the index does not change.

We will show that $N$ is even and therefore $q$ is an integer. When $\lambda = 0$ Eq. (3.1) has a $2n$-tuple multiplier $\rho = 1$. Since $H(t) - D > 0$, as $\lambda$ increases the multipliers of the first kind move along the unit circle in opposite directions and can only converge after encountering multipliers of a different kind [4]. The values of $\lambda$ for which $\rho_k = 1$ are the eigenvalues of problem (3.1). We will assume for simplicity that the multiplicity of this multiplier is 2 for all $\lambda_p$ (this can be achieved by as small a perturbation of the Hamiltonian as desired, without affecting the value of $N$). If this pair of multipliers $\rho_k$ and $\rho_{k+n} = 1/\rho_k$ meet at the point $\rho = 1$, moving along the unit circle, the multiplicity $q_p$ of the corresponding eigenvalue $\lambda_p$ is equal to 2 or 1. In this first case, the multipliers continue moving along the circle, while in the second case they are shifted along the real axis [6]. If $\rho_k$ and $\rho_{k+n}$ meet at the point $\rho = 1$, moving along the real axis, then $q_p = 1$; these multipliers then move along the circle. As is well known, only four multipliers ($\rho_k$, $\rho_k^*$, $1/\rho_k$ and $1/\rho_k^*$) can fall on the positive semiaxis and coincide with it, passing the point $\rho = 1$. Taking these facts into account and bearing in mind that when $\lambda = 1$ there are no multipliers on the real axis (the corresponding Eq. (3.1) is strongly stable), we obtain that the number $N$ of eigenvalues $\lambda_k$ in (0, 1) is even.

We will show that the quantity $q$ is equivalent to the above index $k$ of the stability region, i.e. $q_1 \neq q_2$ when $k_1 \neq k_2$ and $q_1 = q_2$ when $k_1 = k_2$. Suppose $q_1 \neq q_2$, $H(t, s)$ is a continuous curve connecting Hamiltonians $H_1(t)$ and $H_2(t)$. As was shown above, an increase in the number $r$ does not change the index, and hence we can assume that $H(t,s) > D$ when $s \in [0, 1]$. Taking into account the continuity of the eigenvalues $\lambda_k(s)$ and the inequality $N_1 \neq N_2$, we obtain that ($q_1 \neq q_2$) for certain $k$ and $\lambda_k(s) = 1$. The corresponding Eq. (1.1) has an indefinite multiplier $\rho = 1$ and is therefore not strongly stable. Hence, when $q_1 \neq q_2$ the Hamiltonians $H_1(t)$ and $H_2(t)$ cannot be connected by the stable curve $H(t, s)$; consequently, $k_1 \neq k_2$. Conversely, if $k_1 = k_2$, such a curve exists; since the corresponding values $\lambda_k(s) \neq 1$ when $s \in [0, 1]$, we have $q_1 = q_2$. Hence, the quantities $q$ and $k$ are equivalent (besides, it is easy to show that they are equal).

The number of non-zero eigenvalues is independent of $H(t)$, and hence when the Hamiltonian changes the eigenvalues may fall in the interval (0, 1) and can only leave it through the point $\lambda = 1$. Consequently, if with this change $\lambda_k \neq 1$, the index stays the same.

Note that the value of $N$ is equal to the number of zeros $\lambda_k$ in (0, 1) of the real function $\det[X(T, \lambda) - I_{2n}] = 0$, where $X(T, \lambda)$ is any matrix of the fundamental solutions of Eq. (3.1). Hence, the index $q$ can be calculated somewhat more simply than the known indices in every case when $H(t) > 0$. For our subsequent analysis, however, it is essential, when using this index, to be able to establish certain properties of the stability regions.

*Note* 2. In practice one often encounters systems described by the Hill vector equation

$$(M(t)\dot{y})^{\cdot} + C(t)y = 0, \quad y \in R^n \tag{3.2}$$

where $M(t)$ and $C(t)$ are symmetric $T$-periodic matrices, where $M(t) > 0$. As is well known, Eq. (3.2) can be reduced to the form (1.1) with $H = H_0(t) = \operatorname{diag}(M^{-1}(t), C(t))$. Suppose $x(t + T) = \rho x(t)$ is the solution of Eq. (3.2) and $|\rho| = 1$; then

$$\int_0^T (H_0 x, x)dt = 2\int_0^T (M\dot{x}, \dot{x})dt - (M\dot{x}, x)\Big|_0^T \tag{3.3}$$

Since $|\rho| = 1$, the term outside the integral is equal to zero; consequently, the left-hand side of (3.3) is positive ($M(t) > 0$). Hence, the above discussion holds when $D = 0$. Hence, when calculating the index of system (3.2) we can take $D = 0$ in (3.1), even in the matrix $C(t)$ and, consequently, $H_0(t)$ is not positive definite.

## 4. DIRECTIONAL CONVEXITY OF THE STABILITY REGIONS

The stability region $\Omega_k$ is said to be directionally convex [4], if it follows from the conditions $H_1(t) \in \Omega_k$ and $H_1(t) \leqslant H_2(t) \in \Omega_k$ that any Hamiltonian $H(t)$ satisfying the inequality

$$H_1(t) \leqslant H(t) \leqslant H_2(t) \tag{4.1}$$

also belongs to this region.

The directional convexity of the stability regions of Eq. (1.1) was established in [7] for certain conditions with respect to the alternation of multipliers of a different kind (these conditions are satisfied when $n = 1$ and $n = 2$, in particular). The following theorem shows that the property of directional convexity in general.

*Theorem* 2. The stability regions of Eq. (1.1) are directionally convex.

*Proof.* We will put $H(t, s) = H_1(t) + s(H_2(t) - H_1(t))$. We will first show that $H(t, s) \in \Omega_k$ when $s \in [0, 1]$. In view of (4.1) $H(t,s)$ increases (does not decrease) with respect to $s$, and hence the corresponding eigenvalue $\lambda_k(s)$ of problem (3.1) decrease [4]. Since $q_1 = q_2(H_1(t), H_2(t) \in \Omega_k)$, we have $\lambda_k(s) \neq 1$ and consequently $\rho_q(s) \neq 1$ when $s \in [0, 1]$, $q = 1, \ldots, 2n$.

We will agree to define sets of multipliers of each kind, apart from a permutation of their indices. Suppose $\rho_k(0)$ and $\rho_p(0)$ are adjacent to multipliers of the first and second kind, respectively, where $\arg \rho_k(0) < \arg \rho_p(0)$ $(0 < \arg \rho_q(0) < 2\pi)$. When $s$ increases the multipliers $\rho_k(s)$ and $\rho_p(s)$ move along the unit circle in opposite directions to one another; we will assume that they meet when $s = s' < 1$. Without loss of generality we will assume that the multiplicity of this multiplier is two (as noted above, this can be achieved for as small a perturbation of $H(t, s)$ as desired). If the corresponding elementary dividers of the matrix are monodromic non-prime, then when $s$ increases further these multipliers converge with the unit circle [4]. As we know, only four multipliers can simultaneously converge and fall in the unit circle, with the exception of the point $\rho = 1$ and $\rho = -1$. For $s = 1$ Eq. (1.1) is stable, and hence, taking into account the condition assumed above regarding the indices of multipliers of the first kind we can assume that $\rho_k(s)$ and $\rho_p(s)$ for certain $s = s' \leqslant 1$ again meet on the same semicircle or (if $\rho_k(s') = \rho_p(s') = -1$) at the point $\rho = -1$ and then continue to move along it in the same directions. If when $s = s'$ these elementary dividers are prime, $\rho_k(s)$ and $\rho_p(s)$ continue to move around the circle [4]. In both cases the relative position of these multipliers on the arc $(0, 2\pi)$ of the circle when $s = 0$ and $s = 1$ is different ($\arg \rho_k(0) < \arg \rho_p(0)$, $\arg \rho_k(1) > \arg \rho_p(1)$).

Using similar discussions we find that if when $s \in [0, 1)$ several encounters of multipliers of a different kind occur on the unit circle, the relative position of each such pair on the circle changes as indicated above. As a result, the multiplier types of the Hamiltonians $H(t, 0)$ and $H(t, 1)$ are different. We can convince ourselves of this by considering the function $\psi(k)$—the difference between the number of multipliers of the first and second kind in the set of the first $k$ multipliers ($k = 1, \ldots, 2n$) (in order of increasing argument). As can be seen, for the change indicated above in the relative position of multipliers of a different kind $\psi(k)$ decreases for certain $k$. Nevertheless, $\psi_1(k) = \psi_2(k)$, since the multiplier types of Hamiltonians $H_1(t)$ and $H_2(t)$ are the same. The contradiction obtained shows that multipliers of a different kind are not encountered as $s$ increases in the interval [0, 1].

We will now assume that Eq. (1.1) with Hamiltonian $H(t)$, which satisfies inequality (4.1), is unstable. We will put $H(t, s) = H_1(t) + s(H(t) - H_1(t))$, in which case, for certain $s \leqslant 1$, multipliers of a different kind $\rho_k(s)$ and $\rho_p(s)$ occur on the unit circle. As shown above, when $H = H_2(t)$ the inequality $\arg \rho_k(s) < \arg \rho_p(s)$ is satisfied for such multipliers when $s \in [0, 1]$. Consequently, in the case considered some of the multipliers $\rho_k(s)$ and $\rho_p(s)$ must traverse a point $\rho_*$ of the unit circle such that

$$\arg \rho_k(1) < \arg \rho_* < \arg \rho_p(1) \tag{4.2}$$

Suppose $x_*(t, s_*)$ is the corresponding solution of $(x_*(t + T, s_*) = \rho_* x_*(t, s_*), s_* < 1)$.
We will consider the boundary-value problem

$$J\dot{\mathbf{x}} = [H_1(t) + \lambda(R(t) - H_1(t))]\mathbf{x}, \quad \mathbf{x}(T) = \rho_* \mathbf{x}(0) \tag{4.3}$$

When $R = H(t, s_*)$ and $\lambda = 1$, Eq. (4.3) has a solution $\mathbf{x}_*(t, s_*)$ and hence $\lambda = 1$ is an eigenvalue. Since $H_2(t) \geqslant H(t) \geqslant H(t, s_*)$, problem (4.3) must have an eigenvalue $\lambda_k \leqslant 1$ when $R = H_2(t)$. Nevertheless, by virtue of (4.2) when $\lambda$ increases in the range [0, 1], the multipliers of Eq. (4.3) do not fall on the point $\rho_*$; hence, there are no eigenvalues in [0, 1]. The contradiction obtained shows that in system (1.1) with Hamiltonian $H(t, s)$ multipliers of a different kind do not occur as $s$ increases in the interval [0, 1]; consequently, $H(t, 1) = H(t) \in \Omega_k$. The theorem is proved.

The practical value of this theorem is as follows. In practice, the Hamiltonian $H(t)$ usually depends on certain parameters ($H = H(t, \mu_1, \ldots, \mu_p)$); the problem consists of finding regions in the space of these parameters which correspond to strong system stability. Numerical and analytic methods have been developed to solve this (see, for example, [2, 8, 9]). The majority of these enable one to find critical values of the parameters corresponding to the boundaries of the stability regions (i.e. coincidence of multipliers of a different kind on the unit circle). In particular, the boundaries of the so-called fundamental stability regions (corresponding to the presence of an indefinite multiplier $\rho = 1$ or $\rho = -1$) are determined from the condition for $T$-periodic or $T$-antiperiodic solutions ($x(t + T) = -x(t)$ to exist in the system, which leads to an equation in the required values of the parameters [8]. With this approach it still remains an open question whether the whole region bounded by the values of the parameters obtained corresponds to strong stability. Theorem 2 enables us to assert that if the Hamiltonian $H(t, \mu_1, \ldots, \mu_p)$ increases with respect to a certain parameter $\mu_q$, the stability regions $\Omega_k$ in the space of the parameters $\mu_1, \ldots, \mu_p$ are convex with respect to $\mu_q$; hence, in this case the answer to the question is in the affirmative, which makes additional calculations unnecessary. The same conclusion obviously holds if $H(t, \mu_1, \ldots, \mu_p)$ increases with respect to $\mu_q$ (it is sufficient to change to a new parameter $-\mu_q$).

In problems of parametric oscillations and the dynamic stability of linear systems, the Hamiltonian, as a rule, can be represented in the form $H(\omega t, \varepsilon)$, where $\omega$ and $\varepsilon$ are the frequency and intensity of the parametric excitation, in the plane of which the stability regions are also constructed. Here usually $H(\omega t + 2\pi, \varepsilon) = H(\omega t, \varepsilon) > 0$, $H(\omega t, 0) = H_0$ is a constant Hamiltonian. Assuming $\tau = \omega t$, we can write Eq. (1.1) in the form

$$Jx' = H(\tau, \varepsilon, \omega)x \tag{4.4}$$

$$H(\tau, \varepsilon, \omega) = H(\tau, \varepsilon)/\omega, \quad H(\tau + 2\pi, \varepsilon) = H(\tau, \varepsilon), \quad x' = dx/d\tau$$

Since $H_0 > 0$ the matrix $J^{-1}H_0$ has pure imaginary eigenvalues $\pm i\omega_k^0$ ($k = 1, \ldots, n$), where $\omega_k^0$ are the frequencies of free oscillations of the system when $\varepsilon = 0$. Sections separated by the points $\omega_{pk} = (\omega_p^0 + \omega_k^0)/m$ ($p, k = 1, 2, \ldots, n$; $m = 1, 2 \ldots$) correspond to stability regions of system (4.4) on the $\omega$ axis. When $\omega_{pk} = \omega^0{}_p/m$ Eq. (4.4) has a multiplier $\rho = 1$, and hence, from the definition of the index given above, it follows that these points on the $\omega$ axis distinguish stability regions with different indices; the remaining points $\omega_{pk}$ distinguish stability regions which differ solely in the multiplier types.

As can be seen from (4.4) $H(\tau, \varepsilon, \omega)$ decreases with respect to $\omega$ ($H(\tau, \varepsilon) > 0$. Hence, the following assertion follows from Theorem 2.

*Corollary*. The stability regions of system (1.1) when $H(\omega t, \varepsilon) > 0$ are convex with respect to $\omega$.

Hence, by calculating the upper and lower limits $\omega_k^+(\varepsilon)$ and $\omega_k^-(\varepsilon)$ of stability region $\Omega_k$, one can be sure that the system is stable when $\omega \in (\omega_k^-(\varepsilon), \omega_k^+(\varepsilon))$. Note that this result was established for small $\varepsilon$ in [9] by the perturbation method.

*Note* 3. When investigating parametric oscillations in a system described by Hill's vector equation, the latter can usually be written in the form

$$[M(\omega t, \varepsilon)\dot{x}]^{\cdot} + C(\omega t, \varepsilon)x = 0 \tag{4.5}$$

$$M(\omega t, \varepsilon) = M(\omega t + 2\pi, \varepsilon) = M_0 + \varepsilon M_1(\omega t), \quad C(\omega t, \varepsilon) = C(\omega t + 2\pi, \varepsilon) = C_0 + \varepsilon C_1(\omega t)$$

As shown above, when $M(\omega t, \varepsilon) > 0$ for the corresponding Hamiltonian the left-hand side of (3.3) is positive even if the matrix $C(\omega t, \varepsilon)$ is not positive definite. Hence, the stability regions here are convex with respect to $\omega$.

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